

In Search of Exact Solutions

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Many-body systems, such as electrons flowing in a superconductor, are among the most difficult theoretical problems to study. A new family of exactly solvable models may offer some answers.

During the past 40 years, understanding quantum systems involving many particles has been one of the main goals of theoretical physics. This kind of research is designed not to work out the laws of interactions between particles, but, assuming that these interactions are already known, to calculate their effects on a system of n particles. A system where n is large constitutes the ‘many-body problem’, and can exhibit rich behaviour, including phase transitions, superconductivity and Bose-Einstein condensation. The impact of exactly solvable theoretical models on research into these systems is undeniable. With few exceptions, previous exact solutions have applied only to one-dimensional systems. But writing in *Physical Review Letters*, Dukelsky *et. al.* [1] introduce a new family of exactly solvable theoretical models dealing with quantum many-body systems in any number of dimensions: one, two, three or even more.

A wide range of many-body systems has been studied, with varying numbers of particles. For example, systems of nuclei ($n \approx 10^2$ nucleons) are important in nuclear physics, and atomic and molecular physics deal with systems involving $n \approx 10^2$ electrons, whereas mesoscopic ($n \approx 10^2 - 10^6$) and macroscopic ($n \approx 10^{23}$) systems arise in condensed-matter physics. Real systems are usually extraordinarily complex and involve a large set of parameters, most of which are irrelevant for discussing the phenomena under investigation. So theoretical physicists begin by defining a model that is, an idealized system much simpler than the real one, but retaining all the necessary ingredients to discuss the physical properties of interest. However, there are only a few examples of ‘exactly solvable’ models of this type in which one can calculate exactly all the possible quantum states and their energies and/or the various thermodynamic quantities of the system. Such exact solutions always play a decisive role in theoretical physics.

With very few exceptions, the exact solutions of many-body problems can be found only in one dimension. For example, the Tomonaga-Luttinger solution for a one-dimensional system of interacting fermions (particles such as electrons) demonstrates how the Landau-Fermi liquid description of electrons, which has been the theoretical basis of the whole quantum theory of solids, breaks down in one or two dimensions. This discovery had a considerable impact on condensed-matter physics, and gave rise to a new field concerned with the properties of the one-dimensional state known as the ‘Luttinger liquid’. In contrast, several exactly solvable models have been developed in nuclear physics that make use of group-theory techniques. These models are based on the concept of dynamical symmetries, and a non-trivial generalization of this approach is the ‘pairing model’, in which particles that form a pair interact through a force with a particular strength. Such a model has been applied to discussions of superfluidity, and also to studies of superconductivity in small metallic grains.

The pairing model was solved exactly by Richardson [2] more than 30 years ago, a result that passed almost unnoticed despite its enormous potential impact on nuclear as well as condensed-matter physics. In their paper, Dukelsky and colleagues [1] provide an important generalization of Richardson’s work, leading to new classes of pairing models that are exactly solvable in any dimension. To obtain this result, the authors apply group-theoretical and algebraic methods similar to those already used by Gaudin [3]. They discover three classes of integrable (a remarkable property that makes them mathematically tractable) pairing models, which they were able to solve exactly.

In analogy with the work of Gaudin, the three new classes of pairing models found by Dukelsky *et. al.* [1] are named the rational, the trigonometric and the hyperbolic models. The authors go on to apply the rational model to a two-dimensional system of 18 fermions with repulsive interactions, for which they find an exact numerical solution. Their first results indicate that there are attractive correlations between pairs of fermions, despite the underlying repulsive interaction. Such models might apply to high-temperature superconductivity, in which mutually repulsive electrons have to pair up to flow without resistance. In further work, Dukelsky *et. al.* [4] propose an exact correspondence between the solution of the quantum pairing model and a two-dimensional classical electrostatic problem: the trigonometric model can be mapped to an electrostatic problem on a spherical surface, whereas the hyperbolic model seems to correspond to a mapping on a more general surface.

The new solutions of the pairing models seem to be limited to systems of finite size a necessary condition for solving equations giving exact quantum states. But they can certainly be applied to systems where n is quite large, from which the extrapolation to infinite n is fairly accurate. The solutions are also restricted to systems at a temperature of absolute zero (-273° C), but because the exact solution provides all the quantum states and their energies, any thermodynamic quantity can be calculated at a non-zero temperature by thermal averaging, according

to the Boltzmann formula of statistical mechanics. Exact solutions, as well as predicting the particular properties of a simplified system, can also indicate how good a given approximation is to a many-body system.

In all these respects, the models of Dukelsky *et. al.* are highly welcome. Another useful feature of these models is that they apply equally to boson and fermion systems. All known particles are either fermions (such as electrons) or bosons (such as photons), which obey different quantum statistics. As the authors show [1], the rational model may have some bearing on fermion systems, in particular the physics of high-temperature superconductors. In the case of boson systems, Dukelsky and Shuck [5] have applied the rational model to the transition of a group of low-temperature atoms to a Bose-Einstein condensate, a state of matter in which all the atoms share the same quantum state. In similar work, Dukelsky and Pittel [6] applied the model to a system of bosons with short-range repulsive interactions, and obtained new evidence for the validity of the ‘interacting boson model’ of nuclear physics.

The physics of strongly correlated fermion systems is one area of active research that should benefit immediately from these new families of exactly solvable models. Despite considerable theoretical efforts, mainly motivated by the problems raised by high-temperature superconductivity, such systems remain a huge challenge to theorists. These, and other seemingly intractable problems, would welcome any insight offered by a new family of theoretical tools. The results obtained by Dukelsky *et. al.* [1] for a two-dimensional system of fermions with repulsive interactions is a first step in this direction.

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